Representation Theory Review

(following Georgi's *Lie Algebras in Particle Physics* 2^{Ed})

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1 Finite Groups

1.1 Definitions

A *representation*, $D(g)$ of a group G, is a linear space with a mapping from the group elements to a set of operators on the space which satisfy $D(g_1 g_2) = D(g_1)D(g_2)$ and $D(e) = 1$.

A *unitary representation* is one in which all operators are unitary. Two representations, $D_1(g)$ and $D_2(g)$, are *equivalent* iff there is an invertible matrix A such that $D(g) = A^{-1}D(g)A$. A representation is *reducible* if it has an invariant subspace, ie there is a projector P such that $\forall g, D(g)P = PD(g)P$. A representation is *irreducible* if it is not reducible, and a represenation is *completely reducible* if it is equivalent to a direct sum of irreducible represenions.

The *regular representation* is defined by labelling each basis vector of a linear space (with dimension equal to the group order) by one element of the group. The action of $D(g_1)$ on basis vector $|g_2\rangle$ is $D(g_1)g_2 = |g_1 g_2\rangle$.

1.2 Vital Theorems

- Every finite representation is equivalent to a unitary representation.
- Schur's lemma: if two irreducible representations, $D_1(g)$ and $D_2(g)$ satisfy $AD_1(g) = D_2(g)A$, then A is invertible or null. If A is invertible, then D_1 and D_2 are, by definition, equivalent.

  In the case of $D = D_1 = D_2$, A must be proportional to the identity. (ie if an operator commutes with every $D(g)$, it must act trivially upon every element in the space on which D acts.)
- Every representation of a finite group is completely reducible.

1.2.1 Application of Schur's Lemma

Consider an observable O that is invariant under some finite symmetry group which acts as $D(g)$ on the Hilbert space, that is $\forall g, [O, D(g)] = 0$.

If we decompose the Hilbert space into inequivalent irreducible representations $D_a(g)$ of $D(g)$, and label each state as follows: $|a, j, x\rangle$, where a gives which irrep it belongs to, j gives the index within an irrep, and x is any other physical parameter unrelated to the symmetry[1].

Then, consider the matrix elements of O between to any pair of irreps a, b with physical parameters x, y respectively. Let's call this O_{axby}. By the commutation relation above, we gather that $O_{axby} D_a = D_b O_{axby}$. So, by Schur's lemma, we have, for given x, y, the following: if $a \neq b$, then $O_{axby} = 0$, if $a = b$, then O_{axby} is proportional to the identity. This can be written as the following constraint on O:

$$\langle a, j, x| O |b, k, y\rangle = \delta_{a,b}\delta_{j,k} f_a(x, y)$$

So O depends trivially on j, k, and does not connect unlike irreps. All of the physics is contained in $f_a(x, y)$. Note that, if an irrep appears only once, then there is only one possible input to f_a, so the irrep must be an eigenspace of O.

1.2.2 Orthogonality Relations and Counting

By consideration of the following "dyadic" for unitary D_a and D_b,

$$A^{a\,b}_{j\,l} = \sum_g D_a(g^{-1}) |a, j\rangle \langle b, k| D_b(g) \tag{1}$$

we find our first orthogonality relation:

$$\sum_g \frac{n_a}{N} [D_a(g)]^*_{j\,k} [D_b(g)]_{l\,m} = \delta_{a\,b}\delta_{j\,l}\delta_{k\,m} \tag{2}$$

It is simple to write any function of the group elements as a linear combination of the matrix elements (viewed as functions of the group elements) of the regular representation, and this implies that the function could similarly be written as a linear combination of the matrix elements of the irreps. Since we've just shown that the irrep matrix elements are orthogonal, we have that the irrep matrix elements form a complete orthogonal basis for the space of functions $f(g)$. Consequently, the dimension of the spaces of matrix elements of all the irreps is the same as the dimension of the space of functions of the group elements, ie

$$N = \sum_i n_i^2 \tag{3}$$

1.3 Characters

Characters are the traces of the irrep matrices:

$$\chi_D(g) = \mathrm{Tr}\, D(g) \tag{4}$$

Because of the cyclic properties of the trace, (1) equivalent reps have the same character, and (2) for given D, the character is constant on conjugacy classes of G.

[1]Note that the decomposition of D may in general include multiple repititions of the same irrep, D_a. The states of such multiple repititions are distinguished by the value of the "other physical parameter" x. We implicitly take it that, for any a, each repition, D_a has the same matrix elements (ie the matrix elements of D_a do not depend on x).

The orthogonality relation, Eq (2), for irrep matrix elements yields useful orthogonality relations for characters.

$$\frac{1}{N}\sum_{g}\chi_{D_a}(g)^*\chi_{D_b}(g)=\delta_{a\,b} \tag{5}$$

From this, one can show that the characters provide a complete orthogonal basis for functions that are constant over each conjugacy class, and thus the number of irreps equals the number of conjugacy classes. One important case of this is that an abelian group of order N has N one-dimensional irreps.

And the constancy on conjugacy classes can convert the above relation to a sum over conjugacy classes α:

$$\sum_{\alpha}\frac{k_\alpha}{N}\chi_{D_a}(g_\alpha)^*\chi_{D_b}(g_\alpha)=\delta_{a\,b} \tag{6}$$

which implies a complementary orthogonality between characters of conjugacy classes for a sum over representations:

$$\sum_{a}\frac{k_\alpha}{N}\chi_{D_a}(g_\alpha)^*\chi_{D_a}(g_\beta)=\delta_{\alpha\,\beta} \tag{7}$$

Eq (5) can easily be used to find the multiplicity of an irrep in a rep:

$$\frac{1}{N}\sum_{g}\chi_{D_a}(g)^*\chi_{D}(g)=m_a^D \tag{8}$$

because $\chi_{D_a\oplus D_b}=\chi_{D_a}+\chi_{D_b}$. Important case: by considering the character of any regular representation, one can show that it must contain each irrep with a multiplicity equal to that irrep's dimension. This jives well with Eq (3).

We can also find a projector onto any irrep space from Eq (2):

$$P_a=\frac{n_a}{N}\sum_{g}\chi_{D_a}(g)^*D(g) \tag{9}$$

2 Lie Groups

If we have a group smoothly parameterized by some factors α_a, were we chose $g(0)=e$, then we parameterize the representation similarly, and define the group generators[2] $X_a=-i\frac{\partial}{\partial\alpha_a}D(\alpha)$. Then, within a small neighborhood of the identity, $D(d\alpha)=1+id\alpha_aX_a$.

We can define for finite α

$$D(\alpha)=\lim_{k\to\infty}(1+i\alpha_aX_a/k)^k=e^{i\alpha_aX_a}$$

This will define the representation since each the exponentiated term inside the limit approaches the matrix representation of a group element for large k.

[2]This definition ensures that when the representation is unitary, the generators are Hermitian.

Since the exponentials represent the group close to the identity, products of exponentials must themselves be exponentials of generators; that is, for any α and β, there must exist a δ such that

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} = e^{i\delta_d X_d}$$

Expanding and equating powers will show that this requires the generators to form an algebra under commutation:

$$[X_a, X_b] = i f_{abc} X_c$$

The f are called structure factors, and (using that they change sign under $a \leftrightarrow b$, one can show they must be real for unitary representations. The structure constants completely determine the Lie algebra, and do not depend on the representation used to define the generators (they are fixed purely by the group multiplication and smoothness).

The generators will also satisfy the Jacobi identity (which can be seen by expanding out commutators):

$$[X_a, [X_b, X_c]] = [[X_a, X_b], X_c] + [X_b, [X_a, X_c]]$$

2.1 Adjoint Representation

Each group representation furnishes a representation of the Lie algebra by defining the X_a. (A representation of a Lie algebra is a mapping which preserves the commutators).

Alternatively, one natural choice is the adjoint representation which is purely defined by the structure constants. The adjoint representation of the algebra is given by the matrices T_a where

$$[T_a]_{bc} = -i f_{abc}$$

(That this actually represents the Lie algebra can be shown from the Jacobi idenity.)

For a scalar product on the adjoint representation, we may use $\mathrm{Tr}(T_a T_b)$. By a properly chosen linear transform on the X_a (which corresponds to a transform on the α_a used to parameterize the group), we may diagonalize the inner product. That is, we can reparameterize the group in such a way that

$$\mathrm{Tr}(T_a T_b) = k^a \delta_{ab}$$

(where we are still free to rescale the k, but not change their signs). For our purposes, we will care only about algebras in which all the k are positive (*compact* algebras), and rescale such that all are the same value λ. In this basis, the structure factors are completely antisymmetric. In turn, the T_a are Hermitian and the adjoint representation of the group is unitary.

2.2 Simple groups and algebras

An invariant subalgebra is a set of generators which when commuted with any generator in the whole algebra results in a generator in the subalgebra. Exponentiating invariant subalgebras gives invariant subgroups.

An algebra which has no proper invariant subalgebras is called simple. A simple algebra generates a simple group. The adjoint representation of a simple Lie algebra (if chosen as above to be unitary) will be irreducible.

A U(1) subalgebra is a single generator which commutes with all the generators. Since these algebras force a zero value of k^a, the trace product will not give a norm on such spaces. Algebras without U(1) groups are semisimple, and these are constructed from simple algebras. From here on, unless otherwise stated, we deal with compact, semisimple Lie algebras and their unitary representations.

2.3 Transformations

Finite:

$$\begin{aligned} |k\rangle &\to e^{i\alpha_a X_a} |k\rangle \\ \langle k| &\to e^{-i\alpha_a X_a} \langle k| \\ O &\to e^{-i\alpha_a X_a} O e^{-i\alpha_a X_a} \end{aligned}$$

Infinitesimal:

$$\begin{aligned} |k\rangle &\to |k\rangle + \delta |k\rangle, & \delta |k\rangle &= i\alpha_a X_a |k\rangle \\ \langle k| &\to \langle k| + \delta \langle k|, & \delta \langle k| &= - \langle k| i\alpha_a X_a \\ O &\to O + \delta O, & \delta O &= i[\alpha_a X_a, O] \end{aligned}$$

Also, here's a neat formula:

$$\frac{\partial}{\partial \alpha_b} e^{i\alpha_a X_a} = \int_0^1 ds e^{is\alpha_a X_a} (iX_b) e^{i(1-s)\alpha_c X_c}$$

3 SU(2)

The SU(2) algebra is defined by the following commutation relation for the operators J_a where $a = 1, 2, 3$:

$$[J_a, J_b] = i\epsilon_{abc} J_c$$

Here we find all the finite representations of the SU(2) group by the highest-weight construction.

Why only finite representations?

According to the Peter-Weyl theorem, which is beyond our scope, all irreducible Hilbert space representations of a compact group (e.g. SU(2)) are finite dimensional.

3.1 Raising and Lowering Operators

First, we diagonalize J_3. (Since none of the operators commute, diagonalizing one is the best we can do to simplify our lives.) At first, we label the states of the space $|m, \alpha\rangle$ where m is the eigenvalue of J_3 and α distinguishes within degenerate spaces. (We shall see that α is unneccessary later, but for the remainder of this discussion, assume one fixed value of α.)

Define the operators

$$J^{\pm} = (J_1 \pm iJ_2)/\sqrt{2}$$

which satisfy

$$[J_3, J^{\pm}] = \pm J^{\pm}$$
$$[J^+, J^-] = J_3$$

This implies that $J_3 J^{\pm} |m, \alpha\rangle = (m \pm 1) J^{\pm} |m, \alpha\rangle$, so these operators raise and lower the m value by 1. Whatever state in the (a priori degenerate) $m - 1$ eigenspace of J_3 is proportional to $J^- |m, \alpha\rangle$, this is the one we shall label $|m - 1, \alpha\rangle$. (That is, by labelling choice, the lowering operator does not change α from whatever value we have chosen.)

■ **A priori raising and lowering operators** ■

One can make the definition of the raising and lowering operators seem less arbitrary and magical. Take the raising operator for concreteness. In order to have a raising operator J^+ which raises the eigenvalue of J_3, we will want a commutation relation of the following form:

$$[J_3, J^+] = bJ^+$$

where b is some constant. (In fact, it's not neccessary to make b constant; one could walk through a similar discussion with $[J_3, J^+] = J^+ b(J_3)$ which does not a priori assume constant steps.)

Since the algebra is spanned by the J_a, we write $J^+ = \alpha_a J_a$, and the above becomes

$$i\alpha_1 J_2 - i\alpha_2 J_1 = b(\alpha_1 J_1 + \alpha_2 J_2 + \alpha_3 J_3)$$

So, since the J_a are independent, $\alpha_3 = 0$, and we are left with

$$i\alpha_1 = b\alpha_2, \quad \alpha_2 = ib\alpha_1$$

This can only be solved with non-zero α if $b = 1$. The solution is then

$$J^+ \propto (J_1 + iJ_2)$$

Thus we have found the raising operator, up to arbitrary normalization.

So, in the eigenbasis of J_3 the matrix elements of the raising and lowering operator are $[J^{\pm}]_{m'\alpha', m\alpha} = n_m^{\pm}(\alpha)\delta_{m', m\pm 1}$ where $n_m^{\pm}(\alpha)$ is to be determined. Note $n_m^+ = (n_{m+1}^-)^*$, so we may as well just worry about the lowering operator, and say n_m for n_m^-.

Let j be the highest value of m. Since there is no $|j+1,\alpha\rangle$, we must have $J^+|j,\alpha\rangle = 0$. Now we find the matrix elements for the lowering operator by enforcing normalization of the states with lower m values.

$$\begin{aligned} 1 &= \langle j-1,\alpha|j-1,\alpha\rangle \\ &= \frac{1}{|n_j(\alpha)|^2}\langle j,\alpha| J^+J^- |j,\alpha\rangle \end{aligned}$$

So

$$\begin{aligned} |n_j(\alpha)|^2 &= \langle j,\alpha| [J^+,J^-] |j,\alpha\rangle \\ &= \langle j,\alpha| J_3 |j,\alpha\rangle \\ &= j \end{aligned}$$

We are free to set the phase of the states, so we may as well say $n_j(\alpha) = \sqrt{j}$.In fact, doing the same for any $|j-k,\alpha\rangle$ yields a recursion relation for the n_m, which we solve to find

$$n_m(\alpha) = \frac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}$$

Since we are in a finite space, if we keep lowering, we must at some point reach a state such that $J^-|j-l,\alpha\rangle = 0$, that is, $n_{j-l} = 0$. For this to be the case, we must have $j = l/2$ for some integer l. Such a j gives us a finite tower of states (in fact, $2j+1$ states).

Furthermore, since

$$J^-|j-k,\alpha\rangle \propto |j-k-1,\alpha\rangle$$

and

$$J^+|j-k-1,\alpha\rangle \propto J^+J^-|j-k,\alpha\rangle = J_3|j-k,\alpha\rangle \propto |j-k,\alpha\rangle$$

We can lower and raise the states within a tower without changing α. Since J_3 and the raising and lowering operators span the algebra, our set of states labelled by that single α value is invariant. So, since we are considering only irreps, we may discard α, and each J_3 eigenspace is non-degenerate.

We now have the matrix elements of J_3, and $J^\pm$. We found that there is an irrep for any half-integer j (known as the "spin-j" representation). For instance, if we take $j = 1/2$, then we find that $J_a^{1/2} = \frac{1}{2}\sigma_a$, where the σ_a are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This triplet of **S**pecial **U**nitary matrices of dimension **2** is the "defining representation" of SU(2). From here on out, we will switch to a standard notation $|j,m\rangle$ in which, again, m is the J_3 eigenvalue, and j labels the maximum m.

3.1.1 Generalization

The procedure used above works for a general (not necessarily irreducible) representation of SU(2). Diagonalize J_3, find the highest m, and use the lowering operator to find all states in that irrep; then, focus on the space orthogonal to those states just located and recurse. We will later generalize this (the "highest weight decomposition") beyond SU(2).

3.2 Addition of Angular Momentum

If one space $\{|i\rangle\}$ transforms under the representation D_1, and another $\{|j\rangle\}$ under the representation D_2, then the *tensor product* space $\{|i\rangle\,|j\rangle\}$ transforms as $|i\rangle\,|j\rangle \to D_{1\otimes 2}\,|i\rangle\,|j\rangle = (D_1\,|i\rangle)(D_2\,|j\rangle)$. If we examine around the identity, we see that this implies that the representations of the Lie algebras combine in a straightforward manner:

$$J_a^{1\otimes 2} = J_a^1 \otimes I^2 + I^1 \otimes J_a^2$$

or, as we will often abbreviate,

$$J_a^{1\otimes 2} = J_a^1 + J_a^2$$

Since we have diagonalized J_3, this expression becomes particularly easy to work with for J_3 and eigenstates of J_3:

$$J_3^{1\otimes 2}\,|j_1, m_1\rangle\,|j_2, m_2\rangle = (m_1 + m_2)\,|j_1, m_1\rangle\,|j_2, m_2\rangle$$

That is, J_3 values add. One can then use the procedure from Sec 3.1.1 to decompose the representation into a direct sum of spin-j irreps.

Tensor product of spin irreps

The following is Exercise 3.A, but it's important enough to work out here. If we tensor together a spin-j with a spin-s, then the result is a direct sum of representations from $|j-s|$ to $j+s$, stepping by one, ie

$$\{j\} \otimes \{s\} = \bigoplus_{l=|j-s|}^{j+s} \{l\}$$

We prove this by counting the multiplicity of m values in the product and then employing the highest weight decomposition. Let's assume WLOG that $j \geq s$.

- Now, the number of states with $m = j+s$ is one: $\{|j,j\rangle\,|s,s\rangle\}$.
- The number of states with $m = j+s-1$ is two: $\{|j,j-1\rangle\,|s,s\rangle\,,|j,j\rangle\,|s,s-1\rangle\}$.
- And, continuing on, the number of states with $m = j+s-k$ is $k+1$, since we can lower the first factor ket by any number n from 0 to k, and lower the other ket by $k-n$ to ensure that the total is $m = j+s-k$.
- This works until we reach $k = 2s$, at which point we are limited in how many times we can lower the second ket.

In short, we know that, for $k \leq 2s$, there are $k+1$ states with $m = j+s-k$. We could continue counting for $k > 2s$ without much trouble, but we actually won't need to.

Following our procedure, we take the state with the highest m value, $m = j+s$, and lower it, grabbing one state from each m value between $j+s$ and $-(j+s)$. This gives us the spin-$(j+s)$ irrep. Now there is only one state left in $m = j+s-1$, so we grab it, and lower to find the entire spin-$(j+s-1)$ irrep. Now there is only one state left in $m = j+s-2$, etc... From the counting we did above, we see that this gives us all of the irreps between $j+s$ and $j-s$.

Now we can simply count the number of states we've identified to show we've got them all. The total number of states in this representation is $(2j+1)(2s+1)$. For each k between 0 and $2s$, we got an irrep with $2(j+s-k)+1$ states. Some basic arithmetic shows that

$$(2j+1)(2s+1) = \sum_{k=0}^{2s} [2(j+s-k)+1]$$

so we've actually identified all of the states to their irreps already.

4 Tensor Operators

A *tensor operator* is a set of operators which transform under *commutation* with the generators of a Lie algebra the same way that the states of an irrep transform under *multiplication* by those generators, ie $\{O_l\}$ is a tensor operator under an irrep of SU(2) if

$$[J_a, O_l] = O_m [J_a]_{ml}$$

where the J_a are the generators of that Lie algebra in that irrep.

4.1 Choosing the operator basis

For a concrete example, say we have particle with no spin, so that the angular momentum is just orbital: $J_a = L_a = \epsilon_{abc} r_b p_c$. The position operator transforms as

$$[J_a, r_b] = [\epsilon_{acd} r_c p_d, r_b] = -i\epsilon_{acb} r_c = r_c [J_a^{\text{adj}}]_{cb}$$

where "adj" stands for the adjoint representation. The adjoint representation is equivalent to the spin-1 irrep, so a change of basis on the position indices should bring this into "standard form" (ie transforming with the matrix elements of the spin-1 irrep).

■ **$J_a = L_a$?** ■

We didn't actually make the connection yet that J_a equals the orbital angular momentum operator, L_a. In fact, we never said how J_a acts upon this space. One way to go about defining this is to consider what unitary represents rotation on the Hilbert space of 3D spatial wavefunctions, then define J_a as the generator of that unitary and check that it satisfies the proper commutation relations (as in Sakurai).

However, skipping over a lot of detail, we could just say what effect an infinitesimal rotation should have on the position operator. A small rotation of r_b by an angle of α_a about the a axis should add a small component of r_c. That is to say, the infinitesimal change to r_b should be $\delta r_b = \alpha_a \epsilon_{abc} r_c$.

And looking back to the transformation properties of operators in Sec 2.3, we see that an infinitesimal rotation J_a should produce a change $\delta r_b = i[J_a, r_b]$, and thus $[J_a, r_b] = -i\epsilon_{abc} r_c$. Thus we get the commutation relation between the generator of rotation and the position operator.

Since the adjoint representation is equivalent to the spin-1 irrep, we could find the similarity matrix relating the two and use it to transform the r_a operators into standard form. Or we could take a more specific route:

Since $[J_3, r_z] = 0$, we know r_z must be the $m = 0$ operator. And then we just apply the raising and lowering operators $[J^{\pm}, r_z] = \mp(r_x \pm i r_y)/\sqrt{2}$ to find the $m = \pm 1$ operators respectively. So the tensor operator for position in standard form is

$$\{r_0 = r_z, \quad r_{\pm 1} = \mp(r_x \pm i r_y)/\sqrt{2}\}$$

Now there's an important note about decomposing sets of operators which transform under reducible representations. The concept is, again, the highest weight construction, but there's one trick. Whereas with states, one can identify one irrep and then "choose an orthogonal space," we haven't defined a scalar product on the operators. Nonetheless, instead of using orthogonality, we can solve for the linear combination of operators which has a vanishing commutator with J^+. The rest is the same.

4.2 Wigner-Eckart Theorem

The interesting aspect of tensor operators is how the product of a tensor operator and a state transforms:

$$\begin{aligned} J_a O_l^s |j,m,\alpha\rangle &= [J_a, O_l^s] |j,m,\alpha\rangle + O_l^s J_a |j,m,\alpha\rangle \\ &= O_{l'}^s [J_a^s]_{l'l} |j,m,\alpha\rangle + O_l^s |j,m',\alpha\rangle [J_a^j]_{m'm} \end{aligned}$$

that is, the same way as the tensor product of two states. As a first note, we again see that our diagonalized J_3 values add.

$$J_3 O_l^s |j,m,\alpha\rangle = (l+m) O_l^s |j,m,\alpha\rangle$$

Since we already know how product states decompose into irreps, we can use that structure to simplify calculations on tensor operators. Below we will prove the Wigner-Eckart theorem, which gives an expression for the matrix elements of these tensor operators.

Step 1: *We recognize that, because we know the transformation properties of the $O_l^s |j,m,\alpha\rangle$, we must be able to express them in terms of the known basis states with those transformation properties. What that expression is depends, of course, on the operator O_l^s, but all of that dependence is hidden inside the constants $k_{\alpha\beta}$.*

Following the style of the highest-weight decomposition, we know that $O_s^s |j,j,\alpha\rangle$ transforms like the $J_3 = s+j$ state of the spin-$(s+j)$ triplet, so it must be of the form

$$O_s^s |j,j,\alpha\rangle = \sum_\beta k_{\alpha\beta} |s+j, s+j, \beta\rangle$$

for some constants $k_{\alpha\beta} = \langle s+j, s+j, \beta| O_s^s |j,j,\alpha\rangle$. Then, by applying the lowering operator (using the same coefficients we remember from our experience with decomposing tensor product states), we can write out the rest of the spin-$(j+s)$ irrep in terms of the basis states and those $k_{\alpha\beta}$. We could then examine the orthogonal space, and write out the spin-$(j+s-1)$ irrep in terms of the basis states and a different matrix $k_{\alpha\beta}$.

That is precisely what we'll do, but first, a bit of notation for those coefficients. The notation $\langle s,j,l,m|J,M\rangle$ indicates the coefficients which come from this highest weight decomposition of tensor products, specifically the coefficient of the $|s,l\rangle |j,m\rangle$ state in writing out the $J_3 = M$ state of the spin-J irrep. These are known as Clebsch-Gordan coefficients.

$$\sum_l O_l^s |j, M-l, \alpha\rangle \langle s,j,l,M-l|J,M\rangle = \sum_\beta k_{\alpha\beta} |J,M,\beta\rangle \tag{10}$$

The left side contains all those Clebsch-Gordan coefficients which come from group theory, and the right side contains all those $k_{\alpha\beta}$ which come from the specifics of the operator itself.

Step 2: *Figure out how this fixes the matrix elements of our operator.*

We can then invert the above to solve for our object of interest:

$$O_l^s |j,m,\alpha\rangle = \sum_{J=|l-s|}^{l+s} \langle J, l+m | s,j,l,m\rangle \sum_\beta k_{\alpha\beta} |J, l+m, \beta\rangle$$

and take the matrix element

$$\langle J, M, \beta | O_l^s | j, m, \alpha \rangle = \delta_{M,l+m} \langle J, M | s, j, l, m \rangle k_{\alpha\beta}$$

Step 3: *What does it mean?* First of all, the δ function clearly just expresses that J_3 values add. Secondly, we remember we got those $k_{\alpha\beta}$ from just taking matrix elements with the highest-weight state in a given irrep, and we had a different $k_{\alpha\beta}$ for each irrep; ie $k_{\alpha\beta}$ depends on J, but it has no dependence on M, l, m. That dependence is all group theory, ie the values of the matrix elements between different M, l, m can be connected by raising and lowering operators, and that dependence appears inside the Clebsch-Gordan coefficients. So, if we know the value of the $\langle J, M, \beta | O_l^s | j, m, \alpha \rangle$ matrix elements for just one $m+l$ of in a given irrep, we can use our Clebsch-Gordan coefficients to calculate the matrix elements between every other state in that irrep.

That's worth restating: we can calculate $k_{\alpha\beta}$ from any state, and use it to know the rest of the matrix elements in an irrep. That's how strong a constraint the transformation properties place on the operator. In fact, we'll give the $k_{\alpha\beta}$ a new notation to emphasize this: they are the *reduced matrix elements*, written as

$$k_{\alpha\beta} = \langle J, \beta | O^s | j, \alpha \rangle$$

Using the new notation, we write out the Wigner-Eckart theorem as

$$\langle J, M, \beta | O_l^s | j, m, \alpha \rangle = \delta_{M,l+m} \langle J, M | s, j, l, m \rangle \langle J, \beta | O^s | j, \alpha \rangle$$

4.3 Products of Tensor Operators

As one might expect, the products of tensor operators also transform well. In fact, the product of two tensor operators transforms just like the tensor product of two states.

5 Isospin

Is a thing.

6 Roots and Weights

We want to diagonalize as much of an algebra as we can. A subset, $\{H_i\}$, of commuting Hermitian generators which is as large as possible is known as a *Cartan subalgebra*. Because the Cartan generators form a linear space, we can choose a basis in which

$$\mathrm{Tr}(H_i^\dagger H_j) = k_D \delta_{ij}, \quad i, j < m$$

The number, n, of independent generators in the Cartan subalgebra is the *rank*, and, when the whole Cartan subalgebra is simultaneously diagonalized in some representation D, we can write our basis as $|\mu_i, x, D\rangle$, where

$$H_i |\mu_i, x, D\rangle = \mu_i |\mu_i, x, D\rangle$$

ie μ_i is the vector of eigenvalues of the state with respect to the Cartan generators, and x is any other parameter necessary to distinguish the states. The μ_i are called *weights*, and μ is called the *weight vector*.

6.1 The adjoint represenation

We will be able to deduce more from the algebra by studying the adjoint representation, because in this representation, there is the natural correspondence between the states generators and the generators themselves. We will write the state corresponding to any X_a as $|X_a\rangle$, enforce linearity $\alpha X_a + \beta X_b \to \alpha |X_a\rangle + \beta |X_b\rangle$, and choose the scalar product to be

$$\langle X_a | X_b \rangle = \lambda^{-1} \operatorname{Tr}(X_a^\dagger X_b)$$

where λ is the k_D of the adjoint representation.

The algebra acts in a straightforward way upon this representation:

$$X_a |X_b\rangle = |X_c\rangle \langle X_c| X_a |X_b\rangle = |X_c\rangle [T_a]_{cb} =$$
$$-if_{acb} |X_c\rangle = if_{abc} |X_c\rangle = |if_{abc} X_c\rangle = |[X_a, X_b]\rangle$$

In this representation, the weights are also known as *roots*. We see that the states $|H_i\rangle$ will have zero weight vectors, and all states with zero weight vectors must be in the space of these Cartan states. Other states will have non-zero weight vectors. Labelling the generators by the roots of their states:

$$H_i |E_\alpha\rangle = \alpha_i |E_\alpha\rangle$$

■ **Non-degeneracy** ■

Formally, we would need to include another parameter at this point since we haven't shown that the roots uniquely label a state. However, this parameter, like the x in our discussion of SU(2), is a trivial label that would only hang around for a short while. We'll show it to be unnecessary soon enough.

6.2 Raising and lowering

Thus

$$[H_i, E_\alpha] = \alpha_i E_\alpha$$

And note that these E_a are not just the X_a relabelled, but rather are some complex combinations of them selected by the diagonalization of the H_i. So they are not Hermitian; in fact, by conjugation we see that

$$[H_i, E_\alpha^\dagger] = -\alpha_i E_\alpha^\dagger$$

So $E_\alpha^\dagger = E_{-\alpha}$.

Since states with distinct weight vectors disagree in the eigenvalue of a Hermitian, they must be orthogonal. If we make the $|E_\alpha\rangle$ orthonormal, then that fixes the normalization of the E_α, ie

$$\langle E_\alpha | E_\beta \rangle = \lambda^{-1} \operatorname{Tr}(E_\alpha^\dagger E_\beta) = \delta_{\alpha\beta}$$

We see that the E_α are raising/lowering operators of the weight vectors,

$$H_i E_{\pm\alpha} |\mu_i, D\rangle = (\mu_i \pm \alpha_i) E_{\pm\alpha} |\mu_i, D\rangle$$

That expression is true regardless of representation, but, in the case of the adjoint representation, it is particularly important. Since $E_\alpha \left|E_{-\alpha}\right\rangle$ has zero weight, it must be a combination of Cartan states, so $[E_\alpha, E_{-\alpha}]$ is a combination of Cartan generators, ie

$$[E_\alpha, E_{-\alpha}] = \beta_i H_i$$

where

$$\begin{aligned}
\beta_i &= \langle H_i | E_a | E_{-\alpha}\rangle \\
&= \langle [E_\alpha^\dagger, H_i] | E_{-\alpha}\rangle \\
&= \langle \alpha_i E_\alpha^\dagger | E_{-\alpha}\rangle \\
&= \alpha_i \lambda^{-1} \operatorname{Tr}(E_\alpha E_{-\alpha}) \\
&= \alpha_i \lambda^{-1} \operatorname{Tr}(E_{-\alpha}^\dagger E_{-\alpha}) \\
&= \alpha_i
\end{aligned}$$

So

$$[E_\alpha, E_{-\alpha}] = \alpha_i H_i \tag{11}$$

Examining the above commutation relations derived for the $E_{\pm\alpha}$, we see that they are analogous to the $J^\pm$ operators in SU(2).

6.3 SU(2) Sub-algebras

In fact, for every non-zero $\pm\alpha$ pair of roots, we have an SU(2) subalgebra with

$$\begin{aligned}
E^\pm &= |\alpha|^{-1} E_{\pm\alpha} \\
E_3 &= |\alpha|^{-2} \alpha_i H_i
\end{aligned}$$

In general, for any subalgebra, we could decompose the states of the representation into irreps of that subalgebra. Since we already know all of the SU(2) irreps, this heavily constrains the representations of our general algebra.

For instance, we can quickly fill in a detail we skipped over earlier, showing that each distinct root corresponds to exactly one state/generator. By manner of contradition, assume that there are two states $|E_\alpha\rangle$ and $|E'_\alpha\rangle$, which are, without loss of generality, orthogonal. Then

$$\langle E'_\alpha | E_\alpha\rangle = 0 = \lambda^{-1} \operatorname{Tr}(E'_{-\alpha} E_\alpha)$$

Let $\{E^\pm, E_3\}$ be the subalgebra created with E_α. Since $E^- \left|E'_\alpha\right\rangle$ has weight zero, it is a linear combination of the Cartan states. We could find the coefficients of the Cartan states, as done above to get Eq. 11, but at the last step, we find that, rather than having the inner product of E_α with itself, we have the inner product of E_α and E'_α, which is zero by orthogonality. So E^- annihilates $|E'_\alpha\rangle$.

But we also have

$$E_3 \left|E'_\alpha\right\rangle = |\alpha|^{-2} \alpha_i H_i \left|E'_\alpha\right\rangle = \left|E'_\alpha\right\rangle$$

so, one one hand, $|E'_\alpha\rangle$ is a state of $E_3 = 1$. On the other, E^- annihilates $|E'_\alpha\rangle$. This contradicts our findings about SU(2), so it must be that α uniquely specifies a state.

Furthermore, we can show that, if α is a root, then no multiple of it (other than ± 1) is a root. Consider the SU(2) generated by that root. It acts upon the states $\{|E^{\pm}\rangle, |E_3\rangle\}$ as the adjoint representation (equivalent to spin-1). Now we assume by contradiction that $k\alpha$ for some $k \neq \pm 1$ is also a root. Then k must be a half-integer because the E_3 value of any state must be a half-integer.

- If k is an integer, than the state $|E_{k\alpha}\rangle$ can be lowered by E^- until it equals $|E_\alpha\rangle$, but that is a contradiction because $|E_\alpha\rangle$ is already at the top of our spin-1 irrep.
- If k is an half odd integer, then there is a state with root $\beta = \alpha/2$, but then there is a root β and a root 2β, which is disallowed by the previous case.

6.4 Constraining the roots

In any representation D, the E_3 value of a weight is

$$E_3 |\mu, x, D\rangle = \frac{\alpha_i \mu_i}{\alpha^2} |\mu, x, D\rangle$$

Since, for a given root α, we can decompose the representation into SU(2) irreps, and since a weight is an eigenvector of E_3, we can write an arbitrary $|\mu, x, D\rangle$ as a sum of states with the same E_3 value, and each with a definite spin value. Let the highest spin value be j.

We could raise the state $p = j - \frac{\alpha_i \mu_i}{\alpha^2}$ times to annihilate it. Or we could lower it $q = \frac{\alpha_i \mu_i}{\alpha^2} + j$ times. Combining these, we get the "master formula":

$$\frac{\alpha_i \mu_i}{\alpha^2} = -\frac{p-q}{2} \tag{12}$$

This formula is particularly powerful in the adjoint representation, because, if we consider two roots, α and β, and apply the formula to both, we find

$$\frac{\alpha_i \beta_i}{\alpha^2} = -\frac{p-q}{2}, \frac{\beta_i \alpha_i}{\beta^2} = -\frac{p'-q'}{2}$$

Combining the two

$$\frac{(\alpha \cdot \beta)^2}{\alpha^2 \beta^2} = \frac{(p-q)(p'-q')}{4}$$

so the angle between α and β is

$$\cos^2 \theta_{\alpha\beta} = \frac{(p-q)(p'-q')}{4} \tag{13}$$

This is an incredible constraint on the roots, because $(p-q)(p'-q')$ is an integer! And for equality, it must also be between 0 and 4, inclusive. If $(p-q)(p'-q')$ is 4, then $\alpha = \pm\beta$, which is trivial. The other solutions are

$(p-q)(p'-q')$	$\theta_{\alpha\beta}$
0	$\pi/2$
1	$\pi/3$ or $2\pi/3$
2	$\pi/4$ or $3\pi/4$
3	$\pi/6$ or $5\pi/6$

7 SU(3)

Decomposed the adjoint representation in My_All/Projects/RepTheory/SU(3).nb

8 Simple Roots

8.1 Positivity

To generalize our procedure from SU(2), we need a notion of positivity for the weights (so we can talk about raising and lowering unambiguously). The simple definition we will use is that the sign of the weight vector is the sign of its first non-zero entry. This choice is clearly basis dependent, but we will show later that our results are not.

We can then order the weights:

$$\mu > \nu \iff (\mu - \nu \text{ is positive})$$

8.2 Simple Roots

The roots of a generic algebra are not linearly independent. We define the *simple roots* as positive roots which cannot be built from other positive roots. We shall see that, from these roots, we can reconstruct the entire algebra. The following properties are key:

1. If α and β are two different simple roots, then $\alpha - \beta$ is not a root. We see this by contradiction: Say α is larger. Then $\beta = (\alpha - \beta) + (\alpha)$ is a sum of two positive roots, which contradicts our assumption.

2. Because $\alpha - \beta$ is not a root:

$$E_{-\alpha} |E_\beta\rangle = E_{-\beta} |E_\alpha\rangle = 0$$

   We can't lower either state with the other operator, so, in applying the master formula 12, the q for both is zero. Hence

$$\frac{\alpha \cdot \beta}{\alpha^2} = -\frac{p}{2}, \quad \frac{\alpha \cdot \beta}{\beta^2} = -\frac{p'}{2}$$

   Knowing p and p' is equivalent to knowing the angles between and relative lengths of the roots since

$$\cos\theta_{\alpha\beta} = -\frac{\sqrt{pp'}}{2}, \quad \frac{\beta^2}{\alpha^2} = \frac{p}{p'}$$

3. The cosine is negative ($\alpha \cdot \beta$ is negative), so

$$\frac{\pi}{2} \le \theta_{\alpha\beta} < \pi$$

   (The strictness of the latter inequality follows by construction because all simple roots are positive.)

4. The simple roots are thus linearly independent: consider a sum

$$\gamma = \sum_{\alpha} k_\alpha \alpha$$

If all the coefficients are of the same sign, than this cannot vanish, because the value first non-zero entry grows monotonically as terms are summed. But if there are k_α of different signs, we can break the sum up:

$$\gamma = \mu - \nu, \quad \mu = \sum_{k_\alpha > 0} k_\alpha \alpha, \quad \nu = \sum_{k_\alpha < 0} (-k_\alpha)\alpha$$

where μ and ν are both positive. Since

$$\gamma^2 = \mu^2 + \nu^2 - 2(\mu \cdot \nu) > \mu^2 + \nu^2 > 0$$

we see again that γ cannot vanish. (The above holds because $\mu \cdot \nu$ is a sum of the dot products of simple roots, which we showed are all negative quantities.) So the roots are independent.

5. We can thus write any positive root ϕ as a positive sum of simple roots:

$$\phi = \sum_{\alpha} k_\alpha \alpha, \quad k_\alpha > 0$$

(If ϕ is simple, this is trivial. If ϕ is not, then it can be written as a sum of two other positive roots *which must be smaller*, so one could continue down until one arrives at a sum of only simple roots.) In a second we'll see that these k_α must be positive *integers*.

6. Every positive, non-simple root is the sum of a root and a simple root. By contradiction, assume this does not hold for ϕ. Then $E_{-\alpha}\phi = 0$ for all simple α (otherwise there would be a state to which we could apply E_α to get a state with weight ϕ, and we would have our sum). So this ϕ must transform like some combination of the lowest weight states in all the SU(2) algebras, so it's E_3 values $\alpha \cdot \phi / \alpha^2$ must be negative for all α. But since ϕ can be written as a positive sum of simple roots, we have

$$\phi^2 = \sum_{\alpha} k_\alpha \alpha \cdot \phi \leq 0$$

which is a contradiction. This in turn implies that the k_α in the previous point are integers.

7. The simple roots are also complete. Otherwise there would be some vector ξ orthogonal to all the simple roots, so

$$\text{for all } \phi, \quad [\xi \cdot H, E_\phi] = 0$$

but since $\xi \cdot H$ also commutes with the generators, it commutes with the whole algebra, so the algebra is not simple as we assumed. This completeness requires that the number of simple roots equal the rank of the algebra.

8.3 Constructing the roots

Now we can write out a procedure for inductively determining all of the roots from the simple roots. All the positive roots can be written $\phi = \sum_\alpha k_\alpha \alpha$, with k_α positive integers. Call $k = \sum_\alpha k_\alpha$.

Those roots with $k = 1$ are simple roots, so we already know them. Now suppose we have found all the roots with $k \leq l$. Since every positive root is the sum of a positive root (with smaller k) and a simple root, we can find all of the possible roots with $k = l+1$ by adding each simple root to all of the $k = l$ roots. We decide which of those new roots actually exist using the master formula as follows.

Suppose the $k = l$ root was ϕ, and we want to determine whether $\phi + \alpha$ is a root. In

$$\frac{\alpha \cdot \phi}{\alpha^2} = -\frac{(p-q)}{2}$$

we already know the left-hand side. And, because we have all of the smaller roots, we can see how many times it is possible to lower ϕ by α, so we know q. Thus we can find p. If $p > 0$, then $\phi + \alpha$ is a root, otherwise $E_\alpha |E_\phi\rangle$ must be zero.

Roots are often shown with a Dynkin diagram as explained at the bottom of page 111 of the text.

8.4 Constructing the algebra

Now that we have all of the roots, we can determine the entire algebra. We know the commutation relations between the Cartan generators (trivial) and the relations between the Cartan generators and the other generators. We just need to find the relations among those other generators.

For example, suppose we have two simple roots, α and β. If there sum is not a root, we know they commute, and if their sum is a root, we can find the relation as follows:

First, we know that $|[E_\alpha, E_\beta]\rangle = E_\alpha |E_\beta\rangle$ must be an (a priori unnormalized) state with weight $\alpha + \beta$, so it is proportional to $|E_{\alpha+\beta}\rangle$. Second, we know from the determination of the roots where $|E_\beta\rangle$ fits into some α SU(2) irrep, so we know the normalization N of $E_\alpha |E_\beta\rangle$. This fixes the proportionality up to a phase η which we are free to choose:

$$[E_\alpha, E_\beta] = \eta N E_{\alpha+\beta}$$

We make some choice of η, and continue this procedure to write each positive root in terms of commutators of simple roots. Once any generator can be expressed as a commutator of simple roots, any commutation relation can be evaluated in terms of commutation relations of simple roots by using the Jacobi identity.

8.5 Dynkin Coefficients and the Cartan Matrix

In implementing the procedure from Sec 8.3, it is useful to keep track of the $q^i - p^i$ value for each state. This quantity is known as the Dynkin coefficient, and is, by the master formula, twice the E_3 value of the state in the an α^i SU(2) representation. In fact, since

$$q^i - p^i = \frac{2\alpha^i \cdot \mu}{(\alpha^i)^2}$$

and the α^i are linearly independent, the Dynkin coefficient contains the same information as the root vector. Furthermore, the Dynkin coefficients of a general root ϕ can be written in terms of its construction:

$$\begin{aligned} q^i - p^i &= \frac{2\phi \cdot \alpha^i}{(\alpha^i)^2} \\ &= \sum_j k_j \frac{2\alpha^j \cdot \alpha^i}{(\alpha^i)^2} \\ &= \sum_j k_j A_{ji} \end{aligned}$$

Where A is known as the *Cartan matrix*:

$$A_{ji} = \frac{2\alpha^j \cdot \alpha^i}{(\alpha^i)^2}$$

It's easy to see that the diagonal elements of A are all 2's, and the off-diagonal elements record the angles/relative lengths of the simple roots. The j-th row of A is the Dynkin coefficients for the simple root α^j.

When building up all the roots, we just track the Dynkin coefficients and every time we add root α^j, the Dynkin coefficients of the new root are those of the old root plus those of α^j. And since we know the q value from what has already been constructed, we know the p value from the Dynkin coefficient.

Note: the following two rules are generally required to get the p^i and q^i for the simple roots: (1) the p^i value of α^i is zero since $2\alpha^i$ cannot be a root and (2) the q^i value of α^j where $i \neq j$ is zero since $\alpha_j - \alpha_i$ is not a root.

Of course, all of the above is equivalent to just thinking about what SU(2) representation each state will have to fit into since the Dynkin coefficents are just twice the E_3 values.

8.6 Fundamental Weights

The highest weight μ in a representation must be annihilated by all of the positive roots. Since a positive root can be expressed as a multiple commutator of simple roots, we can just say all of the simple roots must annihilate the highest weight of a representation.

$$\forall j, \quad E_{\alpha^j} |\mu\rangle = 0$$

in fact, since we can construct an entire irrep by applying lowering operators, the above is an if and only if.

The Dynkin coefficients l^j of μ must be nonnegative. In fact, every set of l^j gives a μ which is the highest weight of some irrep. It is useful to consider the *fundamental weights* μ^j which satisfy

$$\frac{2\alpha^j \cdot \mu^k}{(\alpha^j)^2} = \delta_{jk}$$

that is, the vector μ^k yields Dynkin coefficients with $l^j = \delta_{jk}$. The irreps formed by lowering the fundamental weights are called *fundamental represenations*, often written as D^j.

■ **Uniqueness and completeness of the fundamental weights** ■

Note: these are unique because of the completeness of the α^j. That is, the defining equation requires that each μ^k is orthogonal to $m-1$ of the α^j, which fixes the direction of each. And the δ normalization then fixes the magnitude of each.

Furthermore, these must be linearly independent: assume there are some coefficients β^k such that $\sum_k \beta^k \mu^k = 0$. Then

$$\forall j, \quad 0 = \sum_k \frac{\alpha^j \cdot \beta^k \mu^k}{(\alpha^j)^2} = \sum_k \beta^k \delta_{jk} = \beta^j$$

so the β^j are all 0, so our μ^k are independent. (And, since there are m of them, they are complete.)

Then any highest weight can be written

$$\mu = \sum_j \mu^j l^j$$

And we can find a representation with that highest weight by taking a tensor product of l^1 copies of the D^1 representation with l^2 copies of the D^2 representation, etc. This will, in general be reducible, but we can find the μ irrep by just lowering that highest weight state.

8.7 Trace of a generator

The trace of a generator of any representation of a compact simple Lie group is zero. since the trace is invariant under similarity transforms, we can just prove this in our standard weight basis:

- All of the raising operators E_α have trace zero because there are no diagonal elements.
- Using the completeness of the α^j, the generators H_i can be written as linear combinations of $\alpha^j \cdot H_j$. But $\alpha^j \cdot H_j$ is proportional to an SU(2) E_3, which, by its symmetry about 0, is traceless.

9 More SU(3)

By example, of SU(3), this chapter further discusses the construction of the representation.

9.1 Constructing the states

Following the highest weight procedure, we know that all states in the irrep can be written

$$E_{\phi_1} E_{\phi_2} \ldots E_{\phi_n} |\mu\rangle$$

where the ϕ_i are roots. But, if any of those ϕ_i are positive roots (ie raising operators), then we could commute them rightward (picking up factors having to do with the weights) until they annihilate the $|\mu\rangle$, so we know we can actually capture all the states using only negative ϕ_i. And since the roots can be broken into simple roots, we further write it

$$E_{-\alpha^{\beta_1}} E_{-\alpha^{\beta_2}} \ldots E_{-\alpha^{\beta_n}} |\mu\rangle$$

where the α^{β_i} are simple roots. From the above form, we see clearly that the highest weight state in an irrep is unique. Furthermore, any weight which can only be acheived via one ordering of the lowering operators is unique (that is, any weight for which there is only one path connecting it to μ).

As for the weights which can be acheived by multiple orderings, how do we determine their uniqueness or degeneracy? Well, with every state written out in this form, it's easy to evaluate inner products via the commutation relations. And taking the inner products allows us to determine whether two states are linearly dependent (ie compare $\langle A|B\rangle \langle B|A\rangle$ against $\langle A|A\rangle \langle B|B\rangle$ and invoke the Cauchy-Schwartz inequality).

9.2 Weyl group

Because SU(2) is symmetric about 0, if μ is a weight, than $\mu - (q^i - p^i)\alpha^i$ is also a weight. Doing this for all weights at the same time has a simple geometric interpretation:

$$\mu - (q^i_\mu - p^i_\mu)\alpha^i = \mu - \frac{2\alpha^i \cdot \mu}{(\alpha^i)^2}\alpha^i$$

The component along α^i is inverted, so this is just a reflection across the hyperplane perpendicular to α^i. This set of reflections (one for each α^i) in the weight space takes the weights to themselves, so it generates a symmetry group, known as the *Weyl group* of the algebra. This sort of symmetry is hinted at by all those hexagonal and triangular plots of roots we keep seeing.

9.3 Complex Conjugation

In the case of SU(3), it turns out that our two fundamental representations have roots which are the negatives of each other. This means the two representations are related by complex conjugation.

Consider the generators T_a of a representation D, with their commutation relations

$$[T_a, T_b] = i f_{abc} T_c$$

Since this can be rewritten

$$[-T_a^*, -T_b^*] = i f_{abc}(-T_c^*)$$

we see that the objects $-T_a^*$ generate a representation $\overline{D}$ of the same algebra (called *the complex conjugate* of the original representation).