Johnson Noise and the Determination of Boltzmann's Constant

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We discuss the phenomenon of Johnson noise and explain how it may be used to measure the value of Boltzmann's constant, then report and analyze an experimental realization of this type of measurement, present the results and detail possible sources of error. This experiment successfully determines Boltzmann's constant via two independent means and also determines the Celsius value of absolute zero. All of these quantities are found to agree with previously reported values to within experimental uncertainty.

I. BACKGROUND AND MOTIVATION

I.1. Boltzmann's Constant and Johnson Noise

Boltzmann's constant, often written k, connects the statistical mechanics of the microscopic world to the thermodynamics of macroscopic scales. It thus generally appears in relations between thermodynamic and microscopic quantities. Consider, for instance, the ideal gas law, $PV = NkT$, where k relates the macroscopic pressure P, volume V, and temperature T of a gas to the microscopic number N of molecules in the gas. As a further example, consider the characteristic voltage of a p-n junction, $V_T = kT/q$, where k relates a voltage V_T and a temperature T to the charge q of a single electron. This property of bridging the division of scales makes k inherently difficult to measure without a means of accessing the microscopic quantites in a macroscopic system (eg counting the number of gas molecules or measuring a single electron's charge). However, one astounding feature of the phenomenon of as Johnson noise is that does allow measurement of Boltzmann's constant via only thermodynamic quantities.

Johnson noise is the universal and unavoidable voltage noise across a resistor at finite temperature, due to thermal agitations in the resistor. The magnitude of this noise is given by the Nyquist thereom (a specific case of the fluctuation-dissapation theorem), and the simplest derivation follows that of Nyquist's original exposition [1], which also provides a great deal of insight into the reason why Johnson noise allows experimental access to Boltzmann's constant.

I.2. Nyquist Theorem

To begin, suppose that two resistors (I and II), both of resistance R, are placed in an otherwise empty circuit, in thermal equilibrium at temperature T. At finite temperature, there will be some thermal agitation of the charges and electromagnetic fields inside each resistor, which will result in a fluctuating voltage across the resistor, and the consequent driving of each resistor by the other. Since the two are in equilibrium, the power supplied from I to the II must be the same as the that from II to I. The following analysis will extract the magnitude of the thermal fluctuations in voltage by deriving that power transfer in two separate ways and equating their expressions: one method invokes the circuit relations, and the other thermodynamics.

For ease of computation, both methods will analyze the fluctuations in an infinitesimal frequency range $d\nu$; we integrate these results over the frequency range of interest afterwards. The first expression for the power follows from basic circuit theory. The average power supplied to II is the product of the voltage and current across it, both of which can be easily derived for such a trivial circuit, modelling the Johnson noise of I as a series voltage supply:

$$\begin{aligned} d\langle P_{II}\rangle &= d\langle I_{II} V_{II}\rangle \\ &= d\langle (V_{II}/2R)(V_{II}/2)\rangle \\ &= d\langle V_{II}^2\rangle/4R \end{aligned}$$

This circuit analysis thus gives the power transferred from I to II.

The thermodynamic expression involves more subtlety. We consider placing between these resistors a single, long (impedance matched) transmission line of length L, with which both are in equilibrium. It is, in fact, this clever thought experiment which explains how Johnson noise grants the experimenter access to Boltzmann's constant: though it is difficult, for instance, to count the number of gas molecules in the ideal gas example, we do know exactly how to enumerate the degress of freedom in a transmission line–these are the electromagnetic modes which satisfy the boundary conditions. Such modes are those with frequencies

$$nv_p/2L, n \in \mathbb{N},$$

where v_p is the propagation velocity of the line. Since each mode contains two degrees of freedom (that is, one electric and one magnetic), the equipartition theorem[1]

[1] Here we assume that we are dealing with frequencies much lower than $kT/\hbar$, which, for room temperature is 6THz. This is well beyond the range of the reported experiment. At these higher frequencies, quantum effects reduce the average energy per mode, such that the total energy converges.

asserts that each mode will contain, on average, an energy kT. From the mode density and average energy, we can determine the energy in an infinitesimal frequency range to be

$$d\langle E\rangle = 2kTL/v_p d\nu.$$

Now, we reinterpret the thermal energy in the modes of the conductor. Each standing wave mode can be seen as a superposition of two travelling waves, one suppling power to each resistor. The average power to II then, is quickly shown to be

$$d\langle P_{II}\rangle = kTd\nu.$$

And we now have a thermodynamic expression to relate to the circuits-based expression. Setting the two powers equal, we arrive at the Nyquist Theorem:

$$d\langle V^2\rangle = 4RkTd\nu$$

The most important feature of this equation for our purposes is that it involves only macroscopic quantities and k. So, because of our knowledge of the electromagnetic modes of a conductor, we should be able to determine Boltzmann's constant with no microscopic measurements.

II. PROCEDURE

II.1. Apparatus and Method

The experimental scheme is depicted in Figure 1. Johnson noise from a resistor (order of microvolts) travels through a twisted pair to a preamp, which amplifies it by a factor of 1000 (to millivolts). The signal is then filtered by a 1kHz-50kHz bandpass, and the RMS voltage is read from a multimeter. Each measurement of voltage involves 15 readings of the multimeter. The switches in the resistor box allow the experimenter to easily short the resistor out (via switch 1) so that noise from the rest of the signal chain can be measured and subtracted out, or reroute the resistor connection (via switch 2) from the signal chain directly to an ohmmeter.

There are two separate methods for determining Boltzmann's constant from this setup. One is to observe the change in Johnson noise levels as the resistance is varied, which simply involves testing a range of resistors (in our case, from 50kΩ to 763kΩ), and a second is to observe the change in Johnson noise as the temperature is varied, which we acheive by inverting the resistor box to insert the resistor into a liquid nitrogen bath or an oven (whose temperature is monitored from a thermocouple held near the resistor). The oven is extemely slow to stabilize at a fixed temperature, so, instead of holding the oven temperature fixed for each measurement, we measure as the temperature drifts and ensure that we collect all fifteen

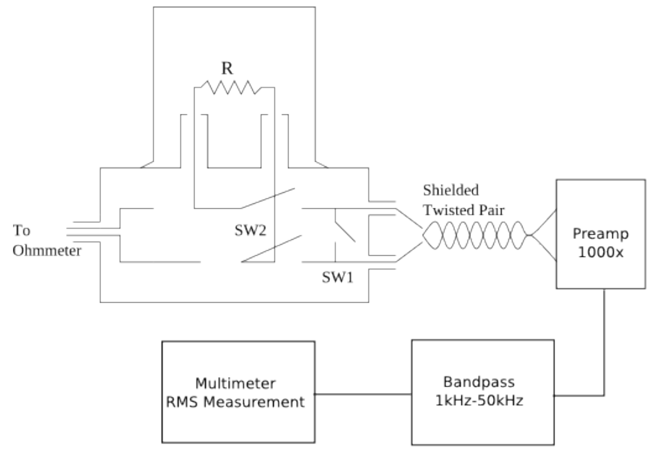

FIG. 1: The noise from the resistor is amplified, filtered, and collected on the multimeter. This image was modified from that appearing here [2].

voltage readings within a range of 1° C about the desired temperature.

In order to actually determine the Johnson voltage across the resistor, we will first require the transfer function from the resistor voltage to the multimeter voltage, as found below.

II.2. Calibrating the Measurement Chain

In determining the transfer function, we divide the signal chain into two parts: that preceeding the preamp, and that including and following the preamp.

The first segment of the chain connects the resistor to the preamp. These wires and the preamp inputs both contribute some capacitance to the circuit. We model this segment as an RC circuit with resistance R (the resistor under study) and a lumped capacitance C, driven by the Johnson noise. This is shown in Figure 2. Using the (complex) voltage divider relation, the voltage at the output (for the frequency ν component of the Johnson noise) is given by

$$d\langle V_{\rm AB}^2\rangle = \frac{1}{1+(2\pi\nu RC)^2} d\langle V_J^2\rangle. \tag{1}$$

That is, the voltage at the preamp is simply the Johnson voltage scaled by a frequency dependent factor, which attenuates higher frequencies. R is known, and C can be determined simply by applying an LCR meter across the resistor clips of the resistor box.

Now, the transfer function of the second segment of the chain (that is, beginning with the preamp) is determined directly. A wavefunction generator (with its output attenuated to microvolts) is connected to the preamp inputs and a separate multimeter. The gain $g(\nu)$ is taken as the ratio of the voltage on the final multimeter to that on this intermediate multimeter, and the transfer function shown in Figure 3 is obtained by varying the frequency on the waveform generator.

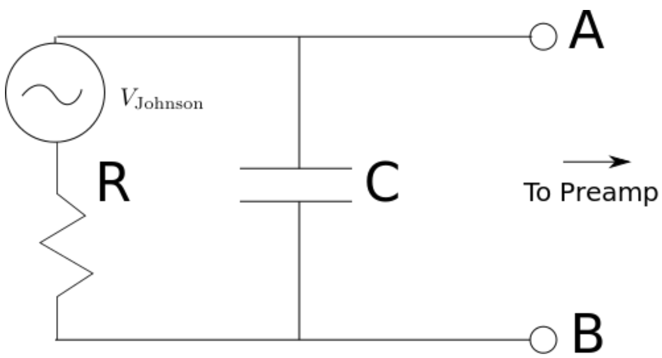

FIG. 2: The first segment of the signal chain constitutes an RC circuit

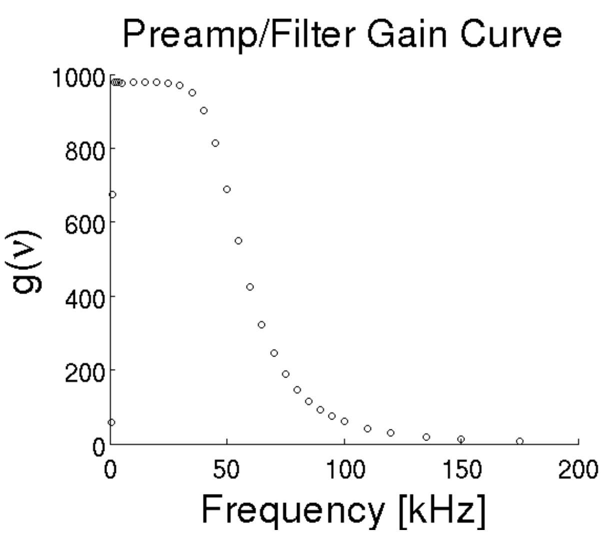

FIG. 3: The transfer function of the second segment of the signal chain is measured directly

Putting these two segments together yields our total transfer function:

$$\langle V^2\rangle d\nu = \frac{g(\nu)^2}{1+(2\pi\nu RC)^2}\langle V_J^2\rangle d\nu$$
$$= \frac{g(\nu)^2}{1+(2\pi\nu RC)^2}4RkTd\nu \quad (2)$$

We integrate up in frequency space in order to determine the RMS voltage which the multimeter will read. Since the only frequency dependence in Equation 2 comes from the transfer function (ie classical Johnson noise is spectrally white), we will just numerically integrate the transfer function and call that factor G:

$$\langle V^2\rangle = 4GRkT \quad (3)$$

And here we have our output voltage in terms of known quantities and k. In practice, there will be some additional Johnson noise from the remainder of the signal chain. To determine the magnitude of this noise, we short out the resistor and measure how much noise $\langle V_S^2\rangle$ remains (this is generally generally about 1mV at output compared to the 4-10mV of output from Johnson noise). Since this noise is non-correlated with the Johnson noise, we will simply subtract out $\langle V_S^2\rangle$ from the mean square voltage on the multimeter to get the actual $\langle V^2\rangle$.

III. DATA AND ANALYSIS

As mentioned, there are two separate procedures for measuring Boltzmann's constant. If we take a sequence of measurements with different resistors at room temperature (25°C), then a plot of $V^2/4GT$ versus R should have a slope of k. The fit is shown in Figure 4. Notice that we also allow a (small) offset term in this fit, as justified below in the error analysis section. This fit estimates $k = (1.383 \pm .013) \times 10^{-23} J/k$, which is consistent with the known value of $k = 1.3807 \times 10^{-23} J/k$.

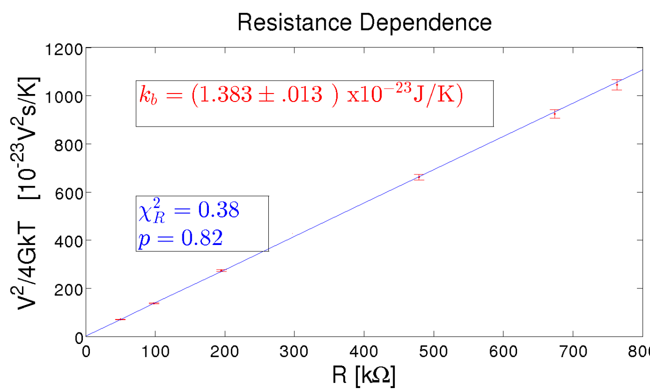

FIG. 4: The scaled Johnson noise is linear in resistance with a slope of Boltzmann's constant.

Alternatively, varying the temperature at constant resistance leads naturally to a plot of $V^2/4GR$ vs T. The slope of such a plot should yield k, and, if we leave the temperatures in Celsius, its extrapolation to the x-axis should yield the Celsius value of absolute zero. The fit, as shown in Figure 5, estimates ($k = 1.387 \pm .019 \times 10^{-23} J/k$ and absolute zero$= 276 \pm 4$°C, both of which are consistent with the known values.

We do note the relatively high χ^2 value, indicating that the points in this plot are subject to more scatter than our error propagation would suggest. The probable cause is unaccounted-for temperature variation (that is, along the horizontal axis), as discussed below.

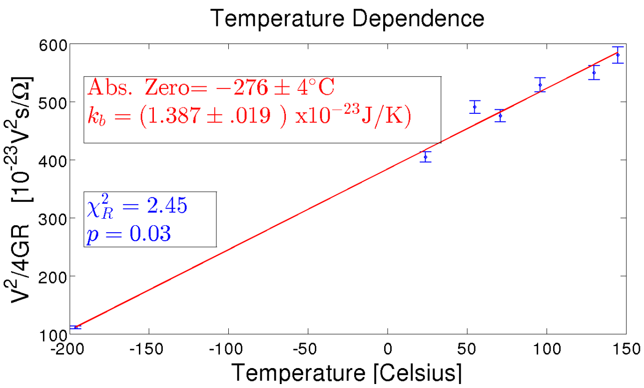

FIG. 5: This scaling of Johnson noise is linear in temperature with a slope of Boltzmann's constant, and an x-intercept at absolute zero.

IV. ERROR ANALYSIS

Several factors allow many of the uncertainties to be kept quite low. For each voltage, fifteen readings were taken from the multimeter, which reduces the statistical uncertainty in $\langle V^2 \rangle$ down below 1%. The resistance values are also easy to measure (and were actually measured at all temperatures involved in the experiment since resistance may vary with temperature), so the uncertainty in R is well below 1% as well. Room temperature was measured by a lab thermometer, and is taken to be accurate to within 1°C throughout the experiment.

However, the uncertainty in capacitance dominates these other statistical errors. The value of C was measured on the day of the resistor-variation measurements to be 60.5pF, and on the day of the temperature-variation measurements to be 61.7pF, both with about 1pF of uncertainty based on the observation that the readout had about 1pF of variation depending on the orienation of the LCR meter. This error propagates through the transfer function to produce about 3% uncertainty in G, which is the main contributor to the errorbars shown in Figures 4 and 5. The uncertainty in k from each of those plots, as given above, is then taken to be the fitting parameter uncertainty from a χ^2 minization. That uncertainty is then further propagated through the linear extrapolation which determines absolute zero.

The x-axis of the temperature-variation plot also contains a systematic uncertainty due to the difficulty of measuring the temperature in the oven. Aside from the previously mentioned 1°C temporal variation (as the oven temperature changes during the course of the 15 rapid measurements), there is another approximately 3°C uncertainty due to spatial variation of temperature within the oven. (The thermocouple was estimated to sit within an inch of the resistor, and the temperature was seen to vary by 3°C when moving the thermocouple by an inch.) This horizontal uncertainty may account for much of the unexpectedly high scatter in the temperature-variation plot.

As a final systemic to discuss, we allow in our resistor-variation fit a small constant offset. This is because the subtraction of $\langle V_S^2 \rangle$ (as discussed in the signal chain) may not perfectly eliminate the additional noise: for instance, removing the resistor changes the RC circuit at the preamp input, which means that the additional noise has a slightly different transfer function when we short the resistor versus when the resistor is present. An additional offset in the fit is justified so long as it remains small. It comes out to roughly 2% of the smallest value of $\langle V^2 \rangle / 4GT$, so we are satisfied that the shorted voltage subtraction is reasonably, but not perfectly, effective for removing additional noise.

V. CONCLUSION

As discussed in the introduction, the known enumeration of electromagnetic modes within a conductor allows the determination of Boltzmann's constant without measurements of any microscopic quantity. Exploiting this phenomenon, we have determined Boltzmann's constant (in two ways) and the Celsius value of absolute zero. Both of these quantities are are consistent to within experimental uncertainty with previously reported values. Future experiments may be able to improve upon this precision further by more detailed study of the capacitance of the input circuit to the preamp.

[1] H. Nyquist, "Thermal Agitation of Electric Charge in Conductors", Phys. Rev., 32, 110-113, (1928).

[2] MIT Physics Junior Lab. http://web.mit.edu/8.13/www/JLExperiments/JLExp43.pdf