Complex Analysis

Sam Bader

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Note of default notation: $z = x + iy, \quad w = u + iv$

1 Analytic Functions

A differentiable complex function of a complex variable is termed *analytic* (or holomorphic or regular or many other names).[1]

1.1 Cauchy-Riemann Equations

$w = u(x, y) + iv(x, y)$ is analytic iff

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

or, equivalently

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial iy}$$

1.2 Interpretations

Geometrical: analytic functions $w(z)$ are *conformal mappings* from the (x, y) plane to the (u, v) plane, ie angles are locally preserved. (That is to say, infinitesmal triangles map onto similar infinitesimal triangles.)

Physical: analytic functions $w = u + iv$ correspond to a vector field $(u, -v)$ that is sourceless and irrotational. Ie. if w is analytic, $\bar{w}$, viewed as a vector field, is sourceless and irrotational.

1.3 Laplace Equation

It follows from the C-R equatoins that both the real and complex parts of an analytic function satisify the Laplace equation, ie

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

1.4 Closure

The set of analytic functions is closed under addition, subtraction, mulitplication, and division (excluding points of division by zero).

[1] A function $w(z)$ can be said to be analytic at infinity if $w(1/z)$ is analytic near $z = 0$.

2 Line Integrals

Interpretation:

$$\int_C w dz = \text{work} + i\,\text{flux}$$

where the vector field is $\bar{w}$.

2.1 Divergence theorem

$$\oint_C w dz = \iint_D \text{div}\ \bar{w} \cdot dxdy + i \iint_D \text{curl}\ \bar{w} \cdot dxdy$$

2.2 Cauchy Theorem

If C bounds a connected region in which w is analytic,

$$\oint_C w dz = 0$$

2.3 Important example: 1/z

Integrate $1/z$ around a circle about the origin

$$\oint_C dz/z$$

2.3.1 Method 1

For a circle

$$dz = izd\theta$$

So

$$\oint_C dz/z = 2\pi i$$

2.3.2 Method 2

In a simply-connected domain, one may use the indefinite integral. We have a discontinuity at $z = 0$, but we can form a simply connected domain by cutting the non-negative real axis. Then we evaluate

$$= \log b - \log a$$

Where this is (arbitrarily) the principal branch of the log function, taken as a goes to 1 and b goes to $e^{2\pi i}$

$$= 2\pi i$$

3 Cauchy Integral Formula

3.1 For $f(z)$

If C is inside a simply-connected domain wherein $f(z)$ is analytic, than for any point z inside C,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(t)dt}{t-z}$$

Thus $f(z)$ inside the domain is fully determined by the values of f on the boundary. This is also a nice trick for evaluating line integrals around non-analytic points.

3.2 For higher derivatives

Furthermore,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(t)dt}{(t-z)^{n+1}}$$

Analytic functions, we see, have all orders of derivatives well-defined.

3.3 Taylor's Theorem

If $f(z)$ is analytic within a disk of radius R about z_0, Taylor's theorem applies within that disk:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

We could write this as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(t)dt}{(t-z_0)^{n+1}}$$

3.4 Laurent's Theorem

If $f(z)$ is analytic within an annulus the same can be done, but negative powers must also be included in the sum:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(t)dt}{(t-z_0)^{n+1}}$$

3.4.1 Singularities and the Residue Theorem

Suppose the $f(z)$ is analytic within some entire domain except possibly at $z = z_0$. Then the negative-exponent, or *principal*, part of the Laurent series describes the nature of the singularity at the $z = z_0$.

1. If all principal terms are zero, then the singularity is removable.
2. If all principal terms after $n = -k$ are zero, but not $n = -k$ itself, we have a pole of order k. (ie $z^k f(z)$ would be analytic, or, at worst, would have a removable discontinuity.)

3. If there are an infinite number of non-zero principal terms, we have an *essential singularity*.

As far as the line integral around the singularity is concerned, only the $n = -1$ term, known as the *residue*, contributes:

$$\oint_C f(z)dz = 2\pi i \cdot a_{-1}$$

Moreover, if a curve encloses some finite number of singularities in an otherwise analytic domain, then the line integral around the curve is

$$\oint_C f(z)dz = 2\pi i \sum \text{residues}$$

This is the Residue Theorem. A common application is to the evaluation of (real) line integrals by taking them as complex line integrals about a semicircle (running along the real axis and curving back at great distance) when the function falls off fast enough at z of large modulus.

3.5 Computation of Residues

1. Removable singularities: residue is zero.

2. First-order poles: the residue at $z = z_0$ is given by

$$\lim_{z \to z_0} (z - z_0) f(z)$$

3. Poles of order k: the residue at $z = z_0$ is given by

$$\frac{1}{(k-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} [(z - z_0)^k f(z)]$$

   In practice, there's generally an easier means, ie for rational functions, one may just recenter the function about the point of interest (that is, rewrite in terms of $t = z - z_0$), Taylor expand the other factors about small t, and read off the coefficient of $1/t$.

4. Essential singularities: my best wishes to you.